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AN AXIOMATIZED FAMILY OF POWER INDICES FOR SIMPLE n-PERSON GAMES

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ABSTRACT

In this probabilistic generalization of the Deegan-Packel power index, new family of power indices based on the notions of minimal winning coalitions and equal division of payoffs is developed. These indices are axiomatically characterized and compared to other similarly characterized indices. Additionally, a dual family of minimal blocking coalition indices and their characterization axioms is presented.

AN AXIOMATIZED FAMILY OF POWER INDICES

FOR SIMPLE n-PERSON GAMES

1. Introduction

In Deegan and Packel (1977) the authors presented a new power index for simple games based upon the ideas of minimal winning coalitions and equal distribution of payoffs among members of such coalitions. In this paper we develop a family of such indices by considering various probability distributions for coalition formation. An axiomatic characterization of these indices is then developed. After comparing our approach with that of Blair (1976) and Dubey (1976), we discuss and rationalize some of the properties and paradoxes which may obtain for our indices. Finally, we develop a family of dual power indices and present the altered axioms which characterize them.

2. Probabilistic Indices and their Characterization

Let $v:2^N \longrightarrow \{0,1\}$, $v(\phi) = 0$, be a simple n-person game on the player set $N = \{1,2,\ldots,n\}$. We assume throughout that v is monotone ($S \subseteq T \Rightarrow v(S) \leq v(T)$), but we do not generally require superadditivity. We denote the class of such games by C_N.

Let $f:2^N \rightarrow (0,\infty)$ be a symmetric ($|S| = |T| \Longrightarrow f(S) = f(T)$) function on the coalitions. For any $v \in C_N$ let $M(v)$ denote the minimal winning coalitions in v. For a given f and v define a probability distribution function $P^f(v)$ on the coalitions in v by

$$P^f(v)(S) = [\alpha^f(v)]^{-1} \begin{cases} 0 & \text{if } S \notin M(v) \\ f(S) & \text{if } S \in M(v) \end{cases}$$

where $\alpha^f(v) = \sum_{S \in M(v)} f(S)$ is the normalization factor for the distribution.

We may then define a probabilistic power index $\rho^f: C_N \rightarrow \mathbb{R}^n$ as follows:

$$\rho_i^f(v) = \sum_{S \ni i} [P^f(v)(S)]v(S)/|S|.$$

ıe rationale for this approach is based upon the modeling assumption that a ›alition S, with a probability $P^f(v)(S)$ of forming, distributes its payoff (S) equally among its $|S|$ members. The fact that $P^f(v)(S)$ is nonzero if and ıly if S is a minimal winning coalition reflects the assumption that only minial winning coalitions can successfully form. Discussion and possible justifiation of these assumptions has been presented in Deegan and Packel (1977) and t greater length in Deegan and Packel (1976). In these earlier works, only he special case where $f \equiv 1$ and $\alpha^f(v) = |M(v)|$ was considered.

We need the following definitions. A player i is a <u>dummy</u> in a game v if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N$ (equivalently, i belongs to no minimal winning coalitions). Players i and j are <u>symmetric</u> in v if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N - \{i,j\}$.

Given v, w ε C_N, we define $v \vee w$ and $v \wedge w$ ε C_N as follows:

$$(v \vee w)(S) = \begin{cases} 1 & \text{if } v(S) = 1 \text{ or } w(S) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$(v \wedge w)(S) = \begin{cases} 1 & \text{if } v(S) = 1 \text{ and } w(S) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

We say that, v,w ε C_N are <u>mergeable</u> if

$$S \in M(v) \text{ and } T \in M(w) \Rightarrow S \not\subseteq T \text{ and } T \not\subseteq S.$$

This mergeability condition is equivalent to requiring that $|M(v \vee w)| = |M(y)| + |M(w)|$ and will be used in the characterization that follows.

<u>Theorem 1</u>. For a given symmetric function $f: 2^N \to (0,\infty)$, the function $\rho^f: C_N \to \mathbb{R}^n$ is the unique power index satisfying the axioms:

A1: $\rho_i^f(v) = 0 \Leftrightarrow$ i is a dummy player.

A2: If i and j are symmetric in v, then $\rho_i^f(v) = \rho_j^f(v)$.

A3: $\sum_{i=1}^{n} \rho_i^f(v) = 1$ for all v ε C_N.

A4: If v and w are mergeable in C_N, then

$$\rho^f(v \vee w) = \frac{\alpha^f(v)\,\rho^f(v) + \alpha^f(w)\,\rho^f(w)}{\alpha^f(v) + \alpha^f(w)}.$$

<u>Proof</u>. Given $\rho_i^f(v) = \sum_{S \ni i} [P^f(v)(S)]v(S)/|S|$, we first show that A1 through A4 are satisfied. Property A1 holds since

$$\begin{aligned} \rho_i^f(v) = 0 &\Leftrightarrow P^f(v)(S)v(S) = 0 \text{ for every } S \text{ containing } i \\ &\Leftrightarrow P^f(v)(S) = 0 \text{ for every } S \in M(v) \text{ containing } i \\ &\Leftrightarrow \text{no } S \in M(v) \text{ can contain } i \\ &\Leftrightarrow i \text{ is a dummy.} \end{aligned}$$

The second axiom is satisfied by the symmetry between i and j and the symmetry of f. For A3 we have

$$\begin{aligned} \sum_{i=1}^{w} \rho_i^f(v) &= \sum_{i=1}^{n} \sum_{S \ni i} [P^f(v)(S)]v(S)/|S| \\ &= \sum_{S \subseteq N} \sum_{i \in S} [P^f(v)(S)]v(S)/|S| \\ &= \sum_{S \subseteq N} P^f(v)v(S) \\ &= \sum_{S \in M(v)} [\alpha^f(v)]^{-1} f(S) = 1. \end{aligned}$$

Finally, A4 holds for ρ^f since, for each i ε N,

$$\rho_i^f(vVw) = \sum_{S \ni i} [P^f(vVw)(S)](vVw)(S)/|S|$$

$$= [\alpha^f(vVw)]^{-1} \Big[\sum_{\substack{S \in M(v) \\ i \in S}} f(S)v(S)/|S| + \sum_{\substack{S \in M(w) \\ i \in S}} f(S)w(S)/|S| \Big]$$

$$= \frac{1}{\alpha^f(v)+\alpha^f(w)} [\alpha^f(v)\rho_i^f(v) + \alpha^f(w)\rho_i^f(w)].$$

ersely, suppose A1-A4 are satisfied by some function $\rho^f: C_N \to \mathbb{R}^n$.
any $v \in C_N$, enumerate the members of $M(v)$ as $S_1, S_2, \ldots, S_m$ and for each .et $v_k \in C_N$ denote the game for which

$$v_k(S) = \begin{cases} 1 & \text{if } S \supseteq S_k \\ 0 & \text{otherwise} \end{cases} .$$

:e $M(v_k) = \{S_k\}$, a singleton set, the v_k are mergeable and $v = v_1Vv_2V\ldots Vv_m$.
any $i \in N$ and any v_k, axioms A1, A2, and A3 require that

$$\rho_i^f(v_k) = \begin{cases} 1/|S_k| & \text{if } i \in S_k \\ 0 & \text{otherwise} \end{cases} .$$

ng A4 readily extended to a merge of m rather than 2 games, we have

$$\rho_i^f(v) = \frac{\sum_{k=1}^m \alpha^f(v_k)\, \rho_i^f(v_k)}{\sum_{k=1}^m \alpha^f(v_k)}$$

$$= [\alpha^f(v)]^{-1} \sum_{k=1}^m \alpha^f(v_k)v_k(S_k)/|S_k|$$

$$= [\alpha^f(v)]^{-1} \sum_{k=1}^m f(S_k)v(S_k)/|S_k|$$

$$= \sum_{S \ni i} [P^f(v)(S)]v(S)/|S|$$

Q.E.D.

3. Examples, Properties, and Paradoxes

The well known power indices of Shapley-Shubik and Banzhaf have been developed (e.g., Blair (1976) or Dubey (1976)) as special cases of what Blair calls P-values in a manner similar to our approach, although their probability distribution P on the coalitions must be formulated in a slightly different manner. More significantly, however, the contributions to the "payoff" of a player i in coalition S are not determined by equal subdivision $[v(S)/|S|]$, but rather by what player i can contribute to the coalition S $[v(S \cup \{i\}) - v(S)]$. In addition, the family of "indices" thus obtained may then require normalization and no coherent set of axioms seems to be available for these normalized indices on C_N.

The differences in assumptions for the Deegan-Packel, Shapley-Shubik, and Banzhaf indices naturally give rise to significantly different models of a priori power determination. While their relative merits might be debated at length it seems likely that, as in so many aspects of n-person game theory, each assumption has a domain of more appropriate applicability. For example, the distribution P in Blair's approach is independent of the game v, while our distribution $P^f(v)$ depends on the coalition structure of the game under consideration. The former approach is somewhat more pleasing in that it generalizes smoothly to provide a family of linear "values" on the class G_N of characteristic function games on N without an efficiency axiom $(\sum_{i=1}^n \rho_i(v) = v(N))$. In contrast, our approach allows incorporation of the nature of each particular game in determining power, while extension to a value on G_N requires a generalization of the minimal winning coalition concept (see Deegan and Packel (1977) for some specialized results in this direction).

Returning to the family of indices defined by

$$\rho_i^f(v) = \sum_{S \ni i} [P^f(v)(S)]\, v(S)/|S|,$$

e that setting $f \equiv 1$ (or any constant value on the minimal winning coalitions) lds the index $\rho: C_N \to \mathbb{R}^n$ defined by

$$\rho_i(v) = |M(v)|^{-1} \sum_{\substack{S \in M(v) \\ i \in S}} 1/|S|$$, and first presented in Deegan and Packel (1976).

s specific index, like those of Shapley-Shubik and Banzhaf, admits a variety "paradoxes" when considered on the domain of weighted voting games. It is : an exercise to demonstrate that the paradoxes of quarreling, added .ght, and new members (e.g., see Brams (1975), Brams and Affuso (1976), . Deegan and Packel (1976)) can be exhibited for ρ. While these paradoxes seem surprising to some, they appear to be present for all power indices may (see Raanen (1976)) in a sense be inevitable.

For the sake of completeness, we feel obliged to point out another :eresting paradox which occurs for ρ but apparently not for the family of iices in Blair's approach. Consider a simple example. The weighted voting ne $v = [5; 3, 2, 1, 1, 1]$ (a quota of 5 with the 5 players having respective ights of 3, 2, 1, 1, and 1) yields power values $\rho(v) = (18/60, 9/60, 11/60,$ $/60, 11/60)$, showing that the "1 vote" players have more power than the vote" player . The impact of this paradox on the interpretation of ρ (and the $v(S)/|S|$ approach) as an a priori measure of power in certain political ting situations is open for discussion. Sociologists have, in fact, argued .g., Caplow (1968)) that situations where minor players possess greater tential for power are not anomalous, but occur rather frequently.

We note, by way of balance, that the indices of Shapley-Shubik and Banzhaf n also be shown to exhibit counter-intuitive properties. In particular, it .s been shown by Straffin (1976) that both the Shapley-Shubik and Banzhaf idices are susceptible to being altered (in the context of weighted voting

games) by players "quarreling" with a dummy player (i.e., refusing to join the same coalition). This cannot happen in our minimal winning coalition approach.

4. Duality

Every $v \in C_N$ gives rise to a natural dual game $v^* \in C_N$ defined as follows:

$$v^*(S) = \begin{cases} 1 \text{ if } v(N-S)=0 \\ 0 \text{ if } v(N-S)=1 \end{cases}.$$

Thus a coalition S is winning ($v^*(S)=1$) in the dual game if and only if it can block any other coalition from winning in the original game v.

We now define a family of dual indices, $\overset{*f}{\rho}: C_N \to \mathbb{R}^n$ by $\overset{*f}{\rho}(v) = \rho^f(v^*)$. Intuitively, such indices measure (for a game v), the power of each player to block winning coalitions from forming (or to prevent passage of motions). It is worth noting that the indices of Shapley-Shubik and Banzhaf are each equal to their duals, thus incorporating the power to initiate and the power to block equally. This is not generally the case for the family of indices we have developed. As a result, our family of indices has the capacity to provide a priori assessments of both "power to initiate" and "power to block," although there are certain situations in which apparently unusual results are obtained.

The family of dual indices can also be characterized axiomatically. The following lemma sets the stage.

Lemma. $(v \wedge w)^* = v^* \vee w^*$ and $(v \vee w)^* = v^* \wedge w^*$.

Proof. $(v \wedge w)^*(S) = 1 \iff (v \wedge w)(N-S) = 0$

$\iff v(N-S) = 0 \text{ or } w(N-S) = 0$

$\iff v^*(S) = 1 \text{ or } w^*(S) = 1$

$\iff (v^* \vee w^*)(S) = 1$

The second equality follows from the first by the reflexive nature ($v^{**} = v$) of the dual.

Q.E.D.

Given $f: 2^N \to (0,\infty)$ symmetric, and $P^f(v)$ a probability distribution on the coalitions of v, we define $\overset{*}{\alpha}{}^f(v) = \alpha^f(v^*)$. Thus $\overset{*}{\alpha}{}^f(v) = \sum_{S \in M(v^*)} f(S)$. We then have:

Theorem 2: The function $\overset{*}{\rho}{}^f: C_N \to \mathbb{R}^n$ is the unique power index satisfying the axioms:

1*: $\overset{*}{\rho}{}_i^f(v) = 0 \iff i$ is a dummy player.

2*: If i and j are symmetric in v, then $\overset{*}{\rho}{}_i^f(v) = \overset{*}{\rho}{}_j^f(v)$.

3*: $\sum_{i=1}^{n} \overset{*}{\rho}{}_i^f(v) = 1$ for all $v \in C_N$.

4*: If v and w have mergeable duals in C_N, then

$$\overset{*}{\rho}{}^f(v \wedge w) = \frac{\overset{*}{\alpha}{}^f(v)\,\overset{*}{\rho}{}^f(v) + \overset{*}{\alpha}{}^f(w)\,\overset{*}{\rho}{}^f(w)}{\overset{*}{\alpha}{}^f(v) + \overset{*}{\alpha}{}^f(w)}$$

Proof. The fact that our $\overset{*}{\rho}{}^f$ satisfies A1* – A3* follows directly from the corresponding results for ρ^f in Theorem 1. It suffices to observe that the sets of dummy players and symmetric pairs in v are unchanged in v*. For A4* we use the Lemma and Theorem 1 as follows.

$$\begin{aligned}\overset{*}{\rho}{}^f(v \wedge w) &= \rho^f((v \wedge w)^*) = \rho^f(v^* \vee w^*) \\ &= \frac{\alpha^f(v^*)\,\rho^f(v^*) + \alpha^f(w^*)\,\rho^f(w^*)}{\alpha^f(v^*) + \alpha^f(w^*)} \\ &= \frac{\overset{*}{\alpha}{}^f(v)\,\overset{*}{\rho}{}^f(v) + \overset{*}{\alpha}{}^f(w)\,\overset{*}{\rho}{}^f(w)}{\overset{*}{\alpha}{}^f(v) + \overset{*}{\alpha}{}^f(w)}\end{aligned}$$

Finally, $\overset{*}{\rho}{}^f$ must be unique in satisfying the axioms on C_N since otherwise the uniqueness of ρ^f established in Theorem 1 would be contradicted.

Q.E.D.

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